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Non-obvious manipulability and efficiency in package assignment problems with money for agents with income effects and hard budget constraints

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Non-obvious manipulability and efficiency in package assignment problems with money for agents with income effects and hard budget constraints*

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Abstract

We study a problem of assigning packages of objects to agents with money. We allow agents to have utility functions that exhibit income effects or face hard budget constraints. It is already known that one of income effects and hard budget constraints lead to the non-existence of a rule satisfying *strategy-proofness*, *efficiency*, *individual rationality*, and *no subsidy* (Dobzinski et al., 2012; Kazumura and Serizawa, 2016; Baisa; 2020, Malik and Mishra, 2021, etc.). Given such negative results, we search for rules satisfying *non-obvious manipulability* (Troyan and Morrill, 2020), an incentive property weaker than *strategy-proofness*, together with the other three properties. First, we identify a necessary and sufficient condition for a rule satisfying *efficiency*, *individual rationality*, and *no subsidy* to be *non-obviously manipulable*. By using the first result, we show that a slight modification of a (truncated) pay as bid rule satisfies *non-obvious manipulability*, *efficiency*, *individual rationality*, and *no subsidy*.

JEL Classification Numbers. D44, D47, D71, D82

Keywords. Non-obvious manipulations, Efficiency, Strategy-proofness, Non-quasi-linear utilities, Hard budget constraints, Package auctions

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1 Introduction

We consider a package assignment problem with money, where a package may involve several objects. We allow each object type to have several copies, or there is only a single object. Examples of our model includes package auctions such as spectrum auctions, airport auctions, privatization auctions, etc., as well as single object auctions. In practical auctions, agents' payments are typically large compared their incomes and budgets, and so both income effects and budget constraints are pervasive (Bulow et al., 2017). We allow agents to have utility functions that exhibit income effects or face hard and private budget constraints.

A (*consumption*) *bundle* is a pair consisting of a package and a payment. An *allocation* specifies each agent's bundle. A *domain* is a class of utility functions. An (*allocation*) *rule* is a mapping from a set of utility profiles to the set of allocations.

A report of a utility function is a *profitable manipulation* if there is other agents' utility functions under which an agent gets better off by reporting it than the truth-telling. A rule is *strategy-proof* if no report of each agent is a profitable manipulation. An allocation *Pareto dominates* another allocation if no agent gets worse off, the revenue of the owner of the objects does not decrease, and the utility of some agent or the revenue gets strictly improved. A rule is *efficient* if an outcome allocation of the rule is not Pareto dominated by any other allocation. A rule is *individually rational* if no agent finds his outcome allocation of the rule worse than receiving no object with the payment of zero. A rule satisfies *no subsidy* if the payment of each agent is always non-negative.

The above four properties are basic requirements, and have been extensively studied in the literature (Chew and Serizawa, 2007; Dobzinski et al., 2012; Kazumura and Serizawa, 2016; Le, 2018; Baisa, 2020; Malik and Mishra, 2021, etc). Unfortunately, it is already known that in a package assignment model, when agents have utility functions that either exhibits income effects or face hard budget constraints, there is no rule that satisfies the four properties (Kazumura and Serizawa, 2016; Baisa, 2020; Malik and Mishra, 2021, Kazumura, 2022; Shinozaki et al., 2022 for income effects; Dobzinski et al., 2012; Le, 2018 for hard budget constraints).

Given impossibility results, there are two typical approaches to further the research. One is to find restricted domains on which there is (not) a rule satisfying the desirable properties (Kazumura and Serizawa, 2016; Baisa, 2020; Malik and Mishra, 2021; Kazumura, 2022; Shinozaki et al., 2022). The other approach is to relax one of the properties (Hafair et al., 2012; Baisa, 2017, 2018; Baisa and Essig Aberg, 2021; Shinozaki, 2023), which we will take in this paper. In particular, we relax *strategy-proofness* to *non-obvious manipulability* (Morrill and Troyan, 2020), a weaker incentive property than *strategy-proofness*, and investigate rules satisfying *non-obvious manipulability*, *efficiency*, *individual rationality*, and *no subsidy*.

In order for a profitable manipulation to be successful, an agent has to know the other agents' utility functions and to understand the details of the rule. However, real-life people often lack such information on other agents' utility functions and ability to engage in contingent reasoning (Li, 2017). This observation leads to the idea that it may be unnecessary for a rule to prevent all possible profitable manipulations, as *strategy-proofness* does.

Troyan and Morrill (2020) introduce the notion of *obvious manipulations* that is easy to recognize and execute successfully. Formally, a profitable manipulation is *obvious* if the best case utility from the manipulation is greater than the best case utility from the truth-telling, or the worst case utility from the manipulation is greater than the worst case utility from the truth-telling. Troyan and Morrill (2020) provide a foundation of obvious manipulations by showing that a profitable manipulation is obvious if and only if even a cognitively limited agent who does not understand the details of a rule can recognize that it is a profitable manipulation. A rule is *not obviously manipulable* if no profitable manipulation is obvious. Thus, a *non-obviously manipulable* rule prevents agents from manipulations that are easy to recognize and execute successfully. Note that *non-obvious manipulability* is weaker than *strategy-proofness*.

First, we derive a necessary and sufficient condition for a rule satisfying *efficiency*, *individual rationality*, and *no subsidy* to be *non-obviously manipulable*. Note that under a rule satisfying *no subsidy*, the best bundle among all the bundles that may be available to a rule (we call it the *best bundle*) is the one at which an agent receives all the objects with the payment of zero.[1] Then, we show that *a necessary and sufficient condition for a rule satisfying efficiency, individual rationality, and no subsidy to satisfy non-obvious manipulability is that, for each agent, the best case utility from the truth-telling is the utility from the best bundle* (Theorem 1).

We exploit a necessary and sufficient condition obtained in Theorem 1 to develop a

[1]We assume both money monotonicity meaning less payments make an agent better off and object monotonicity meaning more objects make him better off.

new rule satisfying *non-obvious manipulability*, *efficiency*, *individual rationality*, and *no subsidy*. To this end, we first find a rule satisfying *efficiency*, *individual rationality*, and *no subsidy*, and then modify it so as to satisfy *non-obvious manipulability* on the basis of Theorem 1.

Note that in our setting of income effects and hard budget constraints, both the valuations of the package (the willingness to pay at the bundle with no object and the payment of zero) and the budget (the ability to pay) are the important information of a utility function. A *truncated valuation* of a package is the smaller of the valuation of the package and the budget, which reflects both the information. A counterpart of a well-known pay as bid rule in our setting is defined as follows. A rule is a *truncated pay as bid rule* if the packages are allocated to the agents so as to maximize the sum of truncated valuations, and each agent pays the truncated valuation of a package that he obtains. We show that any truncated pay as bid rule satisfies *efficiency*, *individual rationality*, and *no subsidy* (Proposition 1).

A lesson learned from Theorem 1 is that if the best case utility from the truth-telling is equal to the utility from the best bundle, then a rule satisfying *efficiency*, *individual rationality*, and *no subsidy* is *non-obviously manipulable*. Thus, we modify a truncated pay as bid rule so that the best case utility from the truth-telling is the utility from the best bundle. A rule is a *modified* (*truncated*) *pay as bid rule* if whenever there is an agent who has the sufficiently large valuations compared with the other agents, he receives all the objects with the payment of zero and all the other agents receives and pays nothing, and otherwise, its outcome allocation is equivalent to that under a truncated pay as bid rule.[2] We establish that *any modified pays as bid rule satisfies non-obvious manipulability, efficiency, individual rationality, and no subsidy* (Theorem 2).

It is widely recognized that a (truncated) pay as bid rule is easy to manipulate, and Troyan and Morrill (2020) formalize this idea by showing that it is *obviously manipulable*. An interesting point of Theorem 2 would be that a slight modification of a (truncated) pay as bid rule (a modified pay as bid rule) satisfies a weak incentive property, namely *non-obvious manipulability*.

We claim the advantages of a modified pay as bid rule compared with other standard rules. Indeed, a truncated Vickrey rule (or its modification defined in the same way as a

[2] Note that this definition is informal because we do give the precise meaning of "sufficiently large valuations compared with the other agents". For the formal definition of the rule, see Section 4.3.

modified pay as bid rule) violates *efficiency* (Proposition 2), and so a modified pay as bid rule outperforms a truncated (or modified) Vickrey rule in terms of *efficiency*. A pay as bid rule also outperforms a Warlasian rule (also known as a uniform-price rule) in terms of an application range because a Warlasian rule is often not well-defined due to the non-existence of a Warlasian equilibrium allocation (Gul and Stacchetti, 1999; Baldwin et al., 2023). We will discuss these points in detail in Section 5.

The paper is organized as follows. Section 2 gives a literature review. Section 3 introduces the model. Section 4 presents the main results. Section 5 discusses the advantage of a modified pay as bid rule compared with other standard rules. Section 6 gives concluding remarks. The proofs are relegated to the Appendix.

2 Related literature

2.1 Object allocation problem with money

The fundamental result in package assignment problems with money is that if utility functions are quasi-linear, i.e., if utility functions neither exhibit income effects nor face hard budget constraints, then the Vickrey rules are the only rules satisfying *strategy-proofness*, *efficiency*, *individual rationality*, and *no subsidy* (Holmström, 1979; Chew and Serizawa, 2007). There are two branches of research that attempt to extend this result to non-quasi-linear settings.

The first branch of research considers agents with income effects but without hard budget constraints. In this model, if agents receive several objects as in this paper, then typically no rule satisfies *strategy-proofness*, *efficiency*, *individual rationality*, and *no subsidy* (Kazumura and Serizawa, 2016; Baisa, 2020; Malik and Mishra, 2021; Kazumura, 2022; Shinozaki et al., 2022). In contrast, if there is a single object or agents receive at most one object, then the minimum price Warlasian rules of Demange and Gale (1985) are the only rules satisfying the four properties (Saitoh and Serizawa, 2008; Sakai, 2008; Morimoto and Serizawa, 2015, etc.).

The second branch considers agents with hard budget constraints but without income effects. If agents' budgets are private information as in this paper, no rule satisfies the four properties even in a single object model (Dobzinski et al., 2012; Le, 2018; etc.). Note that the impossibility result holds even in a single object setting, which contrasts the results for agents with income effects but without hard budget constraints. In contrast, if agents' budgets are public information, then the clinching auctions of Ausubel (2004) are the only rules satisfying the four properties (Dobzinski et al., 2012). Note that in our results we require a certain domain richness condition that excludes a public budget case.

This paper is motivated from the impossibility results in both the branches of research, and belongs to both. A contribution of this paper to both the branches is to obtain a positive existence result by relaxing *strategy-proofness* to *non-obvious manipulability* (Theorem 2) in a general model that includes both the income effects model and the hard budget constraints model as specical cases

Several authors have studied the consequences of relaxing (or dropping) one of the four properties to escape from impossibility results (Hafalir et al., 2012; Baisa, 2017, 2018; Baisa and Essig Aberg, 2021; Shinozaki, 2023). Hafalir et al. (2012), Baisa (2017, 2018), and Baisa and Essig Aberg (2021) show that once we weaken *strategy-proofness* to a weak incentive property such as Nash implementation (Hafalir et al., 2012; Baisa, 2017), implementation in iterated elimination of weakly dominated strategies (Baisa, 2018), or implementation in undominated strategies (Baisa and Essig Aberg, 2021), we can find rules satisfying their respective weak incentive properties, *efficiency* (or a weak efficiency property in Hafalir et al. (2017) and Baisa and Essig Aberg (2021)), *individual rationality*, and *no subsidy*. This paper complements this line of research by studying a consequence of relaxing *strategy-proofness* to *non-obvious manipulability*, a weak incentive property different from theirs. Also, Shinozaki (2023) shows the possibility of a rule satisfying *strategy-proofness*, *efficiency*, and *individual rationality* by giving up *no subsidy* instead of relaxing *strategy-proofness*.

2.2 Non-obvious manipulability

Since Troyan and Morrill (2020) introduced the concept of obvious manipulations, several authors have investigated *non-obvious manipulability* in various models (Aziz and Lam, 2021; Arribillaga and Bonifacio, 2024 for a voting model; Ortega and Segal-Halevi, 2022 for a cake-cutting model; Psomas and Verma, 2022 for a probabilistic assignment model without money; Troyan, 2022 for an assignment model without money; Arribillaga and Risma, 2023 for a matching model with contracts, Archbold et al., 2023a,b for a social choice model with money for quasi-linear utilities).

Notably, Troyan and Morrill (2020) study *non-obvious manipulability* in package assignment problems with money, and show that in an identical objects model with quasi-linear utilities, a minimum uniform-price rule (also known as a minimum price Warlasian rule) is *non-obviously manipulable*, while a pay as bid rule is *obviously manipulable*. Note that for quasi-linear utilities (and as we will show for utilities with income effects and hard budget constraints in Propositions 1 and 3), both a minimum uniform-price rule and a pay as bid rule satisfy *efficiency*, *individual rationality*, and *no subsidy*. The first result of this paper (Theorem 1) extends their results by deriving a necessary and sufficient condition for a rule satisfying the three properties to be *non-obviously manipulable*, and indeed, it implies their results as corollaries.

Troyan and Morrill (2020) and Archbold et al. (2023a) study a bilateral trade model which is closely related to a package assignment model with money, and show that no rule satisfies *non-obvious manipulability*, *efficiency*, *individual rationality*, and *weak budget balance*. A positive result in this paper (Theorem 2) is in contrast to their negative results.

3 Model

There are $n \geq 2$ agents and $m \geq 1$ types of objects. Let $N = \{1, \ldots, n\}$ denote the set of agents, and $M = \{1, \ldots, m\}$ the set of object types. Each object type $a \in M$ has $q^a \in \mathbb{N}$ copies. Let $q = (q^a)_{a \in M}$. A package of objects that an agent $i \in N$ receives is $x_i \in (\mathbb{N} \cup \{0\})^m$. Let $X = \{x_i \in (\mathbb{N} \cup \{0\})^m : x_i \leq q\}$.[3] Let $\mathbf{0} = (0, \ldots, 0) \in X$. A payment that an agent $i \in N$ makes is $t_i \in \mathbb{R}$. The (**consumption**) **set** of an agent $i \in N$ is $X \times \mathbb{R}$. A (**consumption**) **bundle** of an agent $i \in N$ is a pair $z_i = (x_i, t_i) \in X \times \mathbb{R}$. Our model includes a single object allocation model ($m = 1$ and $q = 1$), a multi-unit object allocation model ($m = 1$ and $q \geq 2$), and a combinatorial object allocation model ($m \geq 2$ and $q = (1, \ldots, 1)$) as special cases.

An agent $i \in N$ has a utility function $u_i : X \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ such that for some $b_i \in \mathbb{R}_{++} \cup \{\infty\}$, we have that for each $z_i = (x_i, t_i) \in X \times \mathbb{R}$, if $t_i \leq b_i$, then $u_i(z_i) > -\infty$, and if $t_i > b_i$, then $u_i(x_i, t_i) = -\infty$. Note that for each $x_i \in X$, $u_i(x_i, 0) > -\infty$. For notational convenience, for each $x_i \in X$, let $u_i(x_i, \infty) = -\infty$.

[3]Throughout the paper, we employ the following notations on the vector inequalities: For each pair $x, y \in \mathbb{R}^m$, $x \geq y$ if and only if $x^a \geq y^a$ for each $a \in M$, and $x > y$ if and only if $x \geq y$ and $x \neq y$, i.e., $x^a \geq y^a$ for each $a \in M$ with at least one strict inequality.

We may impose the following properties on a utility function.

Money monotonicity. For each $x_i \in M$ and each pair $t_i, t'_i \in \mathbb{R}$ with $t_i < t'_i \leq b_i$, we have $u_i(x_i, t_i) > u_i(x_i, t'_i)$.

Object monotonicity. For each pair $x_i, x'_i \in X$ with $x_i > x'_i$ and each $t_i \in \mathbb{R}$ with $t_i \leq b_i$, we have $u_i(x_i, t_i) > u_i(x'_i, t_i)$.

Finiteness. For each $z_i \in X \times \mathbb{R}$ and each $x_i \in X$, if $u_i(z_i) \geq u_i(x_i, b_i)$, then there is $t_i \in \mathbb{R}$ such that $u_i(z_i) \leq u_i(x_i, t_i)$.

Continuity. For each $x_i \in X$, $u_i(x_i, \cdot)$ is continuous on $(-\infty, b_i]$.[4]

Let $\overline{\mathcal{U}}$ denote the class of all utility functions satisfying the above four properties. Our generic notation for a class of utility functions satisfying the four properties is $\mathcal{U}$. Thus, $\mathcal{U} \subseteq \overline{\mathcal{U}}$. We call $\mathcal{U}$ a **domain**.

Given $u_i \in \mathcal{U}$, $z_i \in X \times \mathbb{R}$, and $x_i \in X$ with $u_i(z_i) \geq u_i(x_i, b_i)$, by finiteness and continuity, there is a payment $t_i \in \mathbb{R}$ such that $u_i(z_i) = u_i(x_i, t_i)$. By money monotonicity, such a payment is unique. We call the unique payment t_i such that $u_i(z_i) = u_i(x_i, t_i)$ the **valuation** of x_i at z_i for u_i, and denote it by $V_i(x_i, z_i)$. Thus, $u_i(z_i) = u_i(x_i, V_i(x_i, z_i))$. For notational convenience, if $u_i(z_i) < u_i(x_i, b_i)$, then let $V_i(x_i, z_i) = \infty$.

We introduce the four special classes of utility functions that have been studied extensively in the literature.

- A utility function u_i is **quasi-linear** if there is a function $v_i : X \to \mathbb{R}_+$ such that (i) $v_i(\mathbf{0}) = 0$, (ii) for each pair $x_i, x'_i \in X$ with $x_i > x'_i$, $v_i(x_i) > v_i(x'_i)$, and (iii) for each $(x_i, t_i) \in X \times \mathbb{R}$, $u_i(x_i, t_i) = v_i(x_i) - t_i$. Let $\mathcal{U}^Q$ denote the class of all quasi-linear utility functions. Note that if $u_i \in \mathcal{U}^Q$, then $b_i = \infty$, i.e., an agent faces no hard budget constraint. Note also that if $u_i \in \mathcal{U}^Q$, then for each pair $x_i, x'_i \in X$ and each $t_i \in \mathbb{R}$, $V_i(x_i, (x'_i, t_i)) - t_i = v_i(x_i) - v_i(x'_i)$. Thus, in particular, for each $x_i \in X$, $V_i(x_i, (\mathbf{0}, 0)) = v_i(x_i)$.

- A utility function u_i **faces (only) soft budget constraints** if there is a (soft) budget

[4] We only require that $u_i(x_i, \cdot)$ be left-continuous at $t_i = b_i$.

$\beta_i \in \mathbb{R}_{++} \cup \{\infty\}$, an interest rate $r \in \mathbb{R}_{++}$, and a function $v_i : X \to \mathbb{R}_+$ such that (i) $v_i(\mathbf{0}) = 0$, (ii) for each pair $x_i, x_i' \in X$ with $x_i > x_i'$, $v_i(x_i) > v_i(x_i')$, and (iii) for each $(x_i, t_i) \in X \times \mathbb{R}$,

$$u_i(x_i, t_i) = \begin{cases} v_i(x_i) - t_i & \text{if } t_i \leq \beta_i, \\ v_i(x_i) - \beta_i - (1+r)(t_i - \beta_i) & \text{if } t_i > \beta_i. \end{cases}$$

Let $\mathcal{U}^{SB}$ denote the class of all utility functions that face soft budget constraints. Note that if $u_i \in \mathcal{U}^{SB}$, then $b_i = \infty$. Note also that $\mathcal{U}^Q \subsetneq \mathcal{U}^{SB}$.

- A utility function u_i **faces** (**only**) **hard budget constraints** if there is a function $v_i : X \to \mathbb{R}_+$ such that (i) $v_i(\mathbf{0}) = 0$, (ii) for each pair $x_i, x_i' \in X$ with $x_i > x_i'$, $v_i(x_i) > v_i(x_i')$, and (iii) for each $(x_i, t_i) \in X \times \mathbb{R}$,

$$u_i(x_i, t_i) = \begin{cases} v_i(x_i) - t_i & \text{if } t_i \leq b_i, \\ -\infty & \text{if } t_i > b_i. \end{cases}$$

  Let $\mathcal{U}^{HB}$ denote the class of all utility functions that face hard budget constraints. Note that $\mathcal{U}^Q \subsetneq \mathcal{U}^{HB}$. Note also that $\mathcal{U}^{HB} \cap \mathcal{U}^{SB} = \mathcal{U}^Q$.

- A utility function u_i **exhibits** (**only**) **income effects** if $b_i = \infty$, i.e., hard budget constraints are never binding. Let $\mathcal{U}^{IE}$ denote the class of all utility functions that exhibit income effects. Note that $\mathcal{U}^Q \subsetneq \mathcal{U}^{IE}$ and $\mathcal{U}^{SB} \subsetneq \mathcal{U}^{IE}$. Note also that $\mathcal{U}^{IE} \cap \mathcal{U}^{HB} = \mathcal{U}^Q$.

A (**feasible**) **object allocation** is an n-tuple $x = (x_i)_{i \in N} \in X^n$ such that $\sum_{i \in N} x_i \leq q$. Let $\mathcal{X}$ denote the set of object allocations. A (**feasible**) **allocation** is an n-tuple $z = (z_i)_{i \in N} = (x_i, t_i)_{i \in N} \in (X \times \mathbb{R})^n$ such that $(x_i)_{i \in N} \in \mathcal{X}$.

A **utility profile** is an n-tuple $u = (u_i)_{i \in N}$. Given $u \in \mathcal{U}^n$ and $i \in N$, let $u_{-i} = (u_j)_{j \in N \setminus \{i\}}$.

An (**allocation**) **rule on** $\mathcal{U}^n$ is a function $f : \mathcal{U}^n \to Z$. Given a rule f on $\mathcal{U}^n$, let $x^f : \mathcal{U}^n \to \mathcal{X}$ denote the object allocation rule associated with f, and $t^f : \mathcal{U}^n \to \mathbb{R}^n$ the associated payment rule. Given a rule f on $\mathcal{U}^n$, $u \in \mathcal{U}^n$, and $i \in N$, let $f_i(u) = (x_i^f(u), t_i^f(u))$ denote an outcome bundle of agent i for u under f.

We introduce the two incentive properties of rules. A report of a utility function $u_i' \in \mathcal{U}$ is a **profitable manipulation** under a rule f on $\mathcal{U}^n$ at $u_i \in \mathcal{U}$ if there is $u_{-i} \in \mathcal{U}^{n-1}$ such that $u_i(f_i(u_i', u_{-i})) > u_i(f_i(u_i, u_{-i}))$. The following property requires no profitable manipulation exist under a rule.

Strategy-proofness. For each $u \in \mathcal{U}^n$, each $i \in N$, and each $u_i' \in \mathcal{U}$, $u_i(f_i(u)) \geq u_i(f_i(u_i', u_{-i}))$.

Morrill and Troyan (2020) introduce the concept of *obvious manipulations*. In words, a report u_i' is an obvious manipulation of a rule f on $\mathcal{U}^n$ at u_i if either the best case utility from the manipulation is greater than that from the truth-telling, or the worst case utility from the manipulation is greater than that from the truth-telling. Formally, a report $u_i' \in \mathcal{U}$ is an **obvious manipulation** of a rule f on $\mathcal{U}^n$ at $u_i \in \mathcal{U}$ if we have either[5]

$$\sup_{u_{-i} \in \mathcal{U}^{n-1}} u_i(f_i(u_i', u_{-i})) > \sup_{u_{-i} \in \mathcal{U}^{n-1}} u_i(f_i(u_i, u_{-i})),$$

or

$$\inf_{u_{-i} \in \mathcal{U}^{n-1}} u_i(f_i(u_i', u_{-i})) > \inf_{u_{-i} \in \mathcal{U}^{n-1}} u_i(f_i(u_i, u_{-i})),$$

The following incentive property is introduced by Morrill and Troyan (2020), which requires that no obvious manipulation exist under a rule.

Non-obvious manipulability. For each $i \in N$ and each pair $u_i, u_i' \in \mathcal{U}$, we have

$$\sup_{u_{-i} \in \mathcal{U}^{n-1}} u_i(f_i(u_i, u_{-i})) \geq \sup_{u_{-i} \in \mathcal{U}^{n-1}} u_i(f_i(u_i', u_{-i}))$$

and

$$\inf_{u_{-i} \in \mathcal{U}^{n-1}} u_i(f_i(u_i, u_{-i})) \geq \inf_{u_{-i} \in \mathcal{U}^{n-1}} u_i(f_i(u_i', u_{-i})).$$

Note that *strategy-proofness* implies *non-obvious manipulability*.

[5]The original definition of an obvious manipulation in Morrill and Troyan (2020) compares the *maximum* utility from a manipulation with the *maximum* utility from the truth telling in a model with finite outcomes, and the *minimum* utilities. Because our model involves infinite outcomes (allocations), the maximum utilities and the minimum utilities may not be well-defined. In order to avoid such technical difficulties, as in Ortega and Segal-Halevi (2022), we employ the supremum and the infimum operators. Note that such a change of definition does not change the main results in this paper essentially. Indeed, even if we employ the original definition of *non-obvious manipulability* in Troyan and Morrill (2020), with a minor technical condition, the main results still hold.

Given $u \in \mathcal{U}^n$ and a pair of allocations $z = (z_i)_{i \in N} = (x_i, t_i)_{i \in N} \in Z$ and $z' = (z'_i)_{i \in N} = (x'_i, t'_i)_{i \in N} \in Z$, z **Pareto dominates** z' for u if (i) for each $i \in N$, $u_i(z'_i) \geq u_i(z_i)$, (ii) $\sum_{i \in N} t'_i \geq \sum_{i \in N} t_i$, and (iii) for some $i \in N$, $u_i(z'_i) > u_i(z_i)$, or $\sum_{i \in N} t'_i > \sum_{i \in N} t_i$.

The following property requires that a rule should select an allocation that is not Pareto dominated by any other allocation.

Efficiency. For each $u \in \mathcal{U}^n$, there is no $z \in Z$ that Pareto dominates $f(u)$ for u.

The following property is a participation constraint, which requires that each agent should find his outcome bundle of a rule at least as desirable as $(\mathbf{0}, 0)$.

Individual rationality. For each $u \in \mathcal{U}^n$ and each $i \in N$, $u_i(f_i(u)) \geq u_i(\mathbf{0}, 0)$.

Finally, the following property requires that the payment of each agent be non-negative.

No subsidy. For each $u \in \mathcal{U}^n$ and each $i \in N$, $t_i^f(u) \geq 0$.

The following fact is already known in the literature (Dobzinski et al., 2012; Kazumura and Serizawa, 2016; Le, 2018; Baisa, 2020; Malik and Mishra, 2021, Kazumura, 2022; Shinozaki et al., 2022).

Fact 1. *(i) Let $\sum_{a \in M} q^a \geq 2$. Let $\mathcal{U} \supseteq \mathcal{U}^{IE}$. No rule on $\mathcal{U}^n$ satisfies strategy-proofness, efficiency, individual rationality, and no subsidy.*
(ii) Let $\mathcal{U} \supseteq \mathcal{U}^{HB}$. No rule on $\mathcal{U}^n$ satisfies strategy-proofness, efficiency, individual rationality, and no subsidy.

Fact 1 motivates us to give up one of the properties. In this paper, we relax *strategy-proofness* to *non-obvious manipulability*, and search for a rule satisfying *non-obvious manipulability*, *efficiency*, *individual rationality*, and *no subsidy*.

4 Main result

In this section, we give the main results of this paper.

4.1 Rich domain

In our main results, we require the following domain richness condition.

Definition 1. A domain $\mathcal{U}$ is **rich** if the following two conditions hold.
(i) For each $\alpha \in \mathbb{R}_{++}$, there is a quasi-linear utility function $u_i \in \mathcal{U} \cap \mathcal{U}^Q$ such that for each pair $x_i, x_i' \in X$ with $x_i > x_i'$, $v_i(x_i) - v_i(x_i') > \alpha$.
(ii) For each $\varepsilon \in \mathbb{R}_{++}$, there is a quasi-linear utility function $u_i \in \mathcal{U} \cap \mathcal{U}^Q$ such that for each $x_i \in X$, $v_i(x_i) < \varepsilon$.

The first condition of richness states that a domain includes a quasi-linear utility function whose valuations are arbitrarily large. The second condition states that a domain includes a quasi-linear utility function whose valuations are arbitrarily small.

Our domain richness condition is so mild that it covers almost all domains of interest. For example, $\overline{\mathcal{U}}$, $\mathcal{U}^Q$, $\mathcal{U}^{SB}$, $\mathcal{U}^{HB}$, and $\mathcal{U}^{IE}$ are all rich. Moreover, the class of all utility functions that exhibit non-negative (or non-positive) income effects, that satisfying the gross (or the net) substitutes condition (Kelso and Crawford,1982; Baldwin et al., 2023), that satisfying the gross (or the net) complements condition (Rostek and Yoder, 2020; Baldwin et al., 2023), that satisfying the single-intersection condition (Gul and Stacchetti, 1999), any superset of one of the beforementioned domains, and any intersection or union of the beforementioned domains are all rich. Note that our domain richness condition does not cover the case of public budgets (Dobzinski et al., 2012; Lavi and May, 2012, etc.).

4.2 Necessary and sufficient condition

The first main of this paper result gives a necessary and sufficient condition for a rule satisfying *efficiency*, *individual rationality*, and *no subsidy* to be *non-obviously manipulable*.

Theorem 1. *Let $\mathcal{U}$ be a rich domain. A rule f on $\mathcal{U}^n$ satisfying efficiency, individual rationality, and no subsidy is non-obviously manipulable if and only if for each $i \in N$ and each $u_i \in \mathcal{U}$, $\sup_{u_{-i} \in \mathcal{U}^{n-1}} u_i(f_i(u_i, u_{-i})) = u_i(q, 0)$.*

To give a sketch of the proof of Theorem 1, we consider the worst case and the best case utilities from a report of a utility function under a rule satisfying *efficiency*, *individual rationality*, and *no subsidy*. Fix an agent $i \in N$, his utility function u_i, and a report u_i'.

First, we consider the worst case utility of agent i from a report u_i'. By richness, we can choose quasi-linear utility functions of the other agents whose valuations are sufficiently large compared with u_i'. Then, *efficiency* implies that he receives no object (Lemma 3 in Appendix A.1), and the combination of *individual rationality* and *no subsidy* implies that he makes the payment of zero (Lemma 1 in Appendix A.1). Thus, his utility is equal to $u_i(\mathbf{0}, 0)$. By *individual rationality*, this corresponds to the worst case utility from the report u_i'. Since u_i' was arbitrary, any report gives the same worst case utility $u_i(\mathbf{0}, 0)$, and thus he cannot improve the worst case utility by reporting a false utility function.

Then, we consider the best case utility from a report u_i'. By richness, we can choose a profile of the other agents' quasi-linear utility functions whose valuations are sufficiently small compared with u_i'. Then, *efficiency* implies that he receives all the objects, i.e., receives q. By *no subsidy*, his payment is no smaller than 0. Thus, by money monotonicity and object monotonicity, the best bundle among all the bundles that may be available to a rule (we call it the *best bundle*) is $(q, 0)$. If the condition in Theorem 1 holds, then the best case utility from the truth-telling is equivalent to the utility from the best bundle $(q, 0)$, and so any report u_i' cannot improve the best case utility from the truth-telling. In contrast, if the condition does not hold, then the best case utility from the truth-telling is smaller than $u_i(q, 0)$, and by reporting a sufficiently small u_i', the best case utility can be improved from the truth-telling, i.e., a rule is *obviously manipulable*.

4.3 Modified pay as bid rule

Theorem 1 does not provide a closed-form characterization of the class of rules satisfying *non-obvious manipulability*, *efficiency*, *individual rationality*, and *no subsidy*. Indeed, the class of rules satisfying *efficiency*, *individual rationality*, and *no subsidy* is so broad that it is difficult to obtain a tractable closed-form characterization of the class of rules satisfying the four properties from Theorem 1. Thus, in this subsection, we instead present a specific rule that satisfies the four properties to show the existence of such a rule.

The design of our new rule satisfying the four properties relies on the two observations: One is the properties of a *truncated pay as bid rule* that we will introduce shortly, and the other is derived from Theorem 1. Thus, we first examine the properties of a truncated pay as bid rule.

Given a utility function $u_i \in \mathcal{U}$ and $x_i \in X$, a **truncated valuation** of x_i (at $(\mathbf{0}, 0)$) for u_i is defined as $\tilde{v}_i(x_i) = \min\{V_i(x_i, (\mathbf{0}, 0)), b_i\}$. A truncated valuation of x_i reflects the valuation $V_i(x_i, (\mathbf{0}, 0))$ of x_i at $(\mathbf{0}, 0)$ and a budget b_i, both of which constitute the important information of a utility function u_i.

A truncated pay as bid rule is defined as follows.

Definition 2. A rule f on $\mathcal{U}^n$ is a **truncated pay as bid rule** if for each $u \in \mathcal{U}^n$,

$$x^f(u) \in \arg\max_{x \in \mathcal{X}} \sum_{i \in N} \tilde{v}_i(x_i),$$

and for each $i \in N$,

$$t_i^f(u) = \tilde{v}_i(x_i^f(u)).$$

The first condition of the above definition says that the packages are assigned to the agents so as to maximize the sum of truncated valuations. The second one says that the payment of each agent is his truncated valuation of a package that he obtains.

The next proposition reveals the desirable properties of a truncated pay as bid rule. Notably, it satisfies *efficiency*. To the best of our knowledge, *efficiency* of a truncated pay as bid rule in the presence of hard budget constraints is a new result.[6]

Proposition 1. *Let $\mathcal{U} \subseteq \overline{\mathcal{U}}$. Any truncated pay as bid rule satisfies efficiency, individual rationality, and no subsidy.*

Troyan and Morrill (2020) show that a truncated pay as bid rule is *obviously manipulable* in a setting of identical objects (which corresponds to the case where $a = 1$) when utility functions are quasi-linear. Although their setting is less general than ours, their argument carries over to any number of object types and any rich domain. Thus, we obtain the following.

Fact 2. *Let $\mathcal{U}$ be rich. Any truncated pay as bid rule is obviously manipulable.*

Recall Theorem 1 shows that to ensure that the best case utility of each agent from the truth-telling is equal to the utility from the best bundle $(q, 0)$ is a necessary and sufficient condition for a rule satisfying *efficiency*, *individual rationality*, and *no subsidy* to be *non-obviously manipulable*. Recall also Proposition 1 states that a truncated pay as bid rule satisfies *efficiency*, *individual rationality*, and *no subsidy*. Thus, once we modify a truncated pay as bid rule so that the best case utility of each agent from the truth-telling is equal to the utility from the best bundle $(q, 0)$, such a rule is *non-obviously manipulable*.

[6] Shinozaki et al. (2022) discuss that a truncated pay as bid rule satisfies the three properties if agents do not face hard budget constraints, but say nothing about the case of hard budget constraints.

We need to modify a pay as bid rule carefully so that the above property holds without damaging its nice efficiency property (and the other desirable properties as well). We modify a truncated pay as bid rule as follows: If there is an agent who has sufficiently large truncated valuations and the valuations at $(q, 0)$ compared with the other agents' truncated valuations, then he receives $(q, 0)$ and all the other agents do $(\mathbf{0}, 0)$ under a rule, and otherwise, an outcome allocation under a rule is equivalent to that under a truncated pay as bid rule.

Definition 3. A rule f on $\mathcal{U}^n$ is a **modified (truncated) pay as bid rule** if for each $u \in \mathcal{U}^n$, if there is $i \in N$ such that for each $x_i \in X \backslash \{\mathbf{0}\}$ and each $x'_i \in X \backslash \{q\}$, $\min\{\tilde{v}_i(x_i), -V_i(x'_i, (q, 0))\} > (n-1) \max_{j \in N \backslash \{i\}} \tilde{v}_j(q)$, then $f_i(u) = (q, 0)$ and $f_j(u) = (\mathbf{0}, 0)$ for each $j \in N \backslash \{i\}$, and otherwise, $f(u)$ is an outcome allocation of a truncated pay as bid rule for u.

Note that the best case utility of each agent from the truth-telling under a modified pay as bid rule is $u_i(q, 0)$. If an outcome allocation under a modified pay as bid rule coincides with that under a truncated pay as bid rule, then it is efficient by Proposition 1. If there is an agent who wins $(q, 0)$ under an outcome allocation of a modified pay as bid rule, then by the definition of the rule, he has so large valuations at $(q, 0)$ compared with the other agents' truncated valuations that such an allocation will be efficient. Thus, a modified pay as bid rule will be *efficient*. Also, *individual rationality* and *no subsidy* of a modified pay as bid rule follow from Proposition 1.[7] Thus, by Theorem 1, it also satisfies *non-obvious manipulability*, and we obtain the following.

Theorem 2. *Let $\mathcal{U}$ be a rich domain. Any modified pay as bid rule on $\mathcal{U}^n$ satisfies non-obvious manipulability, efficiency, individual rationality, and no subsidy.*

5 Discussion

In this section, we show the advantages of a modified pay as bid rule compared with other standard rules.

[7] Note that for each $i \in N$, by money monotonicity and object monotonicity, $u_i(q, 0) > u_i(\mathbf{0}, 0)$.

5.1 Truncated Vickrey rule

First, we compare a modified pay as bid rule with a truncated Vickrey rule which extends a Vickrey rule (Vickrey, 1961) for quasi-linear utility functions to our setting with income effects and hard budget constraints. A rule f on $\mathcal{U}^n$ is a **truncated Vickrey rule** if for each $u \in \mathcal{U}^n$,

$$x^f(u) \in \underset{x \in \mathcal{X}}{\arg\max} \sum_{i \in N} \tilde{v}_i(x_i),$$

and for each $i \in N$,

$$t_i^f(u) = \max_{x \in X} \sum_{j \in N \setminus \{i\}} \tilde{v}_j(x_j) - \sum_{j \in N \setminus \{i\}} \tilde{v}_j(x_j^f(u)).$$

It is well known that a Vickrey rules satisfies *strategy-proofness*, *efficiency*, *individual rationality*, and *no subsidy* when utility functions are quasi-linear (Vickrey, 1961; Chew and Serizawa, 2007). The next proposition states that in the presence of income effects, any truncated Vickrey rule violates *efficiency*, and in the presence of hard budget constraints, it satisfies neither *non-obvious manipulability* nor *efficiency*.

Proposition 2. *(i) Let $\sum_{a \in M} q^a \geq 2$ and $\mathcal{U} = \mathcal{U}^{IE}$. Any truncated Vickrey rule on $\mathcal{U}^n$ satisfies non-obvious manipulability, but violates efficiency.*
(ii) Let $\mathcal{U} = \mathcal{U}^{HB}$. If $\sum_{a \in M} q^a = 1$, then any truncated Vickrey rule satisfies non-obvious manipulability, but violates efficiency. If $\sum_{a \in M} q^a \geq 2$, then any truncated Vickrey rule satisfies neither non-obvious manipulability nor efficiency.

Le (2018) gives an example which shows that a truncated Vickrey rule violates *efficiency* in the presence of hard budget constraints. In the next example, we illustrate that a truncated Vickrey rule violates *efficiency* in the presence of income effects but without hard budget constraints.

Example 1. For simplicity, let $n = 2$, $a = 1$, and $q = 2$, but we can easily extend the following discussion to a more general setting. Let f be a truncated Vickrey rule on $(\mathcal{U}^{IE})^2$. Let $u_1 \in \mathcal{U}^{IE}$ be such that $V_1(1, (0, 0)) = 5$, $V_1(2, (0, 0)) = 6$, and $V_1(2, (1, 2)) = 5$. Let $u_2 \in \mathcal{U}^Q$ be such that for each $x_2 \in M$, $v_2(x_2) = 2x_2$. Then, by the definition of a truncated Vickrey rule, $f_1(u) = (1, 2)$ and $f_2(u) = (1, 1)$. Let $z = (z_i)_{i \in N} = (x_i, t_i)_{i \in N} \in Z$ be an allocation such that $z_1 = (2, 5)$ and $z_2 = (0, -1)$. By $V_1(2, (1, 2)) = 5$, $u_1(z_1) = u_1(2, 5) = u_1(1, 2) = u_1(f_1(u))$. By $u_2 \in \mathcal{U}^Q$, $u_2(z_2) = u_2(0, -1) = 1 = v_2(1) - 1 = u_2(f_2(u))$. Further,

$\sum_{i \in N} t_i = 4 > 3 = \sum_{i \in N} t_i^f(u)$. Thus, z Pareto dominates $f(u)$ for u, and so f violates *efficiency*.

Note that in Proposition 2 (i), we consider the case of at least two objects. In the case of a single object without hard budget constraints, a truncated Vickrey rule satisfies *strategy-proofness* and *efficiency* (Saitoh and Serizawa, 2008; Sakai, 2008).

Theorem 2 and Proposition 2 together suggest that in the presence of income effects, a truncated pay as bid rule overcomes a truncated Vickrey rule in terms of *efficiency*, and in the presence of hard budget constraints, it does a truncated Vickrey rule in terms of *non-obvious manipulability* and *efficiency*.

We omit the detail of the definition, but we can define a *modified Vickrey rule* in the same way as a modified pay as bid rule. Then, we can show that a modified Vickrey rule is *non-obviously manipulable* on any rich domain. However, it still violates *efficiency*. Indeed, in Example 1, we exhibit a utility profile at which both agents 1 and 2 win an objec, and an outcome allocation of a truncated Vickrey rule is not efficient. For such a utility profile, a modified Vickrey rule produces the same outcome allocation as a truncated Vickrey rule since several agents objects, and so a modified Vickrey rule violates *efficiency*. Thus, a modified pay as bid rule outperforms a modified Vickrey rule in terms of *efficiency*. We note that because it is difficult to identify the set of utility profiles at which a truncated Vickrey rule violates *efficiency*, it is unclear how we can achieve both *non-obvious manipulability* and *efficiency* by modifying a truncated Vickrey rule in some way.

5.2 Warlasian rule

Warlasian rules, also known as uniform-price rules in a setting of identical objects (i.e., $a = 1$), are also salient rules in the literature (Kelso and Crawford, 1982; Gul and Stacchetti, 1999; Rostek and Yoder, 2020; Baldwin et al., 2023, etc.). Given $u \in \mathcal{U}^n$, an allocation $(z_i)_{i \in N} = (x_i, t_i)_{i \in N} \in Z$ is a **Warlasian equilibrium allocation** for u if there is a price vector $p \in \mathbb{R}^m_+$ such that for each $i \in N$, $t_i = p \cdot x_i$, and for each $x_i' \in X$, $u_i(z_i) \geq u_i(x_i', p \cdot x_i')$.[8] A rule f on $\mathcal{U}^n$ is a **Warlasian rule** if for each $u \in \mathcal{U}^n$, $f(u)$ is a Warlasian equilibrium allocation for u.

By the first welfare theorem, any Warlasian rule is *efficient*. Further, any Warlasian rule satisfies *individual rationality* and *no subsidy*. However, not all Warlasian rules satisfy

[8]Given a pair $x, y \in \mathbb{R}^m$, $x \cdot y$ is an inner produce ot x and y, i.e., $x \cdot y = \sum_{a \in M} x^a y^a$.

non-obvious manipulability. For example, a pay as bid rule in a single object setting (a *first-price rule*) is a Warlasian rule, but is *obviously manipulable* (Fact 2). The next proposition states that there is a Warlasian rule satisfying *non-obvious manipulability* on a rich domain where there always exists a Warlasian equilibrium allocation.

Proposition 3. *Let $\mathcal{U}$ be a rich domain such that for each $u \in \mathcal{U}^n$, there is a Warlasian equilibrium allocation for u. There is a Warlasian rule on $\mathcal{U}$ that is non-obviously manipulable.*

Thus, in terms of *non-obvious manipulability*, *efficiency*, *individual rationality*, and *no subsidy*, a Warlasian rule is as desirable as a modified pay as bid rule. However, one drawback of a Warlasian rule is that the existence of a Warlasian equilibrium allocation is not necessarily guaranteed, and thus a Warlasian rule may not be well-defined. Indeed, even for quasi-linear utility functions, the existence of a Warlasian equilibrium allocation is no longer guaranteed once a utility function in the domain violates the gross (or the net) substitutes condition (Gul and Stacchetti, 1999; Baldwin et al., 2023). Clearly, both income effects and hard budget constraints further complicate the existence of a Warlasian equilibrium allocation. An advantage of a modified pay as bid rule compared with a Warlasian rule is its broad application range. Indeed, it can be applied to any rich domain that may not guarantee the existence of a Warlasian equilibrium allocation.

6 Conclusion

In this paper, in response to recent impossibility results for *strategy-proofness*, *efficiency*, *individual rationality*, and *no subsidy* for agents with income effects or hard budget constraints, we have relaxed *strategy-proofness* to *non-obvious manipulability*, and studied its implications. As a result, we derived a necessary and sufficient condition for a rule satisfying *efficiency*, *individual rationality*, and *no subsidy* to be *non-obviously manipulable* (Theorem 1), and showed that a modified pay as bid rule satisfies *non-obvious manipulability*, *efficiency*, *individual rationality*, and *no subsidy* (Theorem 2).

We conclude this paper by giving a comment on future research. Recall that we discussed in Section 1 that *strategy-proofness* may be unnecessarily demanding as it requires *all* profitable manipulations be prohibited. Recall also that *non-obvious manipulability* is motivated by an observation that real-life agents often lack the ability to engage in contingent reasoning or do not have the access to the information of the other agents' utilities. Such a situation is plausible if a package auction that involves many objects is conducted via a one-shot sealed-bid format, or the planner does not reveal the detail of the information of the rival bidders. However, practical auctions often involve only a few objects and are conducted via simple open format with the known rival bidders. Then, to study an "intermediate" incentive property between *strategy-proofness* and *non-obvious manipulations* would be an interesting direction of future research. The results and insights derived in this paper will be helpful.

Appendix

A Proof of Theorem 1

In this section, we provide the proof of Theorem 1. Let f be a rule on $\mathcal{U}^n$ satisfying *efficiency*, *individual rationality*, and *no subsidy*.

A.1 The "if" part

We show the "if" part. We begin with the following three lemmas. First, the next lemma shows that if an agent receives no object, then his payment is equal to zero.

Lemma 1. *Let $u \in \mathcal{U}^n$ and $i \in N$. If $x_i^f(u) = \mathbf{0}$, then $t_i^f(u) = 0$.*

Proof. Suppose $x_i^f(u) = \mathbf{0}$. By *individual rationality*, $u_i(\mathbf{0}, t_i^f(u)) = u_i(f_i(u)) \geq u_i(\mathbf{0}, 0)$. Thus, by money monotonicity, $t_i^f(u) \leq 0$. Thus, by *no subsidy*, we get $t_i^f(u) = 0$. □

Given $u_i \in \mathcal{U}$ and $x_i \in X\backslash\{\mathbf{0}\}$, let $h(\cdot; u_i, x_i)$ be a function from $[0, b_i]$ to $\mathbb{R}$ such that for each $t_i \in [0, b_i]$, $h_i(t_i; u_i, x_i) = t_i - V_i(\mathbf{0}, (x_i, t_i))$.[9]

Lemma 2. *Let $u_i \in \mathcal{U}$ and $x_i \in X\backslash\{\mathbf{0}\}$. Then,* $\sup_{t_i \in [0,b_i]} h_i(t_i, u_i, x_i) < \infty$.

Proof. By continuity of u_i, $h_i(\cdot; u_i, x_i)$ is a continuous function on $[0, b_i]$. Since $[0, b_i]$ is compact, $h_i([0, b_i]; u_i, x_i)$ is also compact. Thus, $h_i([0, b_i]; u_i, x_i)$ is bounded, and so $\sup_{t_i \in [0,b_i]} h_i(t_i; u_i, x_i) < \infty$. □

[9]Note that by $x_i \geq \mathbf{0}$ and $t_i \leq b_i$, object monotonicity and money monotonicity together imply $u_i(x_i, t_i) \geq u_i(\mathbf{0}, b_i)$. Thus, $V_i(\mathbf{0}, (x_i, t_i)) < \infty$.

The next lemma states that given a utility function of an agent, there are utility functions of the other agents such that he receives no object.

Lemma 3. *Let $i \in N$ and $u_i \in \mathcal{U}$. There is $u_{-i} \in \mathcal{U}^{n-1}$ such that $x_i^f(u_i, u_{-i}) = \mathbf{0}$.*

Proof. Let $j \in N \backslash \{i\}$. By Lemma 2 and richness, we can choose $u_j \in \mathcal{U} \cap \mathcal{U}^Q$ such that for each $x_i \in X \backslash \{\mathbf{0}\}$ and each pair $x_j, x_j' \in X$ with $x_j > x_j'$,

$$v_j(x_j) - v_j(x_j') > \sup_{t_i \in [0, b_i]} h_i(t_i, u_i, x_i). \tag{1}$$

For each $k \in N \backslash \{i, j\}$, let $u_k = u_j$. We show $x_i^f(u) = \mathbf{0}$. By contradiction, suppose $x_i^f(u) \neq \mathbf{0}$. By *individual rationality*, $t_i^f(u) \leq b_i$. By *no subsidy*, $t_i^f(u) \geq 0$. Thus, we have $t_i^f(u) \in [0, b_i]$. By (1) and $x_i^f(u) \neq \mathbf{0}$,

$$v_j(x_j^f(u) + x_i^f(u)) - v_j(x_j^f(u)) > t_i^f(u) - V_i(\mathbf{0}, f_i(R)). \tag{2}$$

Let $z = (z_k)_{k \in N} = (x_k, t_k)_{k \in N} \in Z$ be an allocation such that $z_i = (\mathbf{0}, V_i(\mathbf{0}, f_i(u)))$, $z_j = (x_j^f(u) + x_i^f(u), t_j^f(u) + v_j(x_j^f(u) + x_i^f(u)) - v_j(x_j^f(u)))$, and for each $k \in N \backslash \{i, j\}$, $z_k = f_k(u)$. Then, $u_i(z_i) = u_i(f_i(u))$. We have $u_j(z_j) = v_j(x_j^f(u)) - t_j^f(u) = u_j(f_j(u))$. We also have

$$\sum_{k \in N} (t_k - t_k^f(u)) = V_i(\mathbf{0}, f_i(u)) - t_i^f(u) + v_j(x_j^f(u) + x_i^f(u)) - v_j(x_j^f(u)) > 0,$$

where the inequality follows from (2). Thus, z Pareto dominates $f(u)$ for u, a contradiction to *efficiency*. □

We move on to the proof of the "if" part. Suppose that for each $i \in N$ and each $u_i \in \mathcal{U}$, we have

$$\sup_{u_{-i} \in \mathcal{U}^{n-1}} u_i(f_i(u_i, u_{-i})) = u_i(q, 0). \tag{3}$$

Let $i \in N$ and $u_i, u_i' \in \mathcal{U}$. For each $u_{-i} \in \mathcal{U}^{n-1}$, by *no subsidy*, $f_i(u_i', u_{-i}) \in X \times \mathbb{R}_+$, and so by money monotonicity and object monotonicity, $u_i(q, 0) \geq u_i(f_i(u_i', u_{-i}))$. Combining this with (3), we get

$$\sup_{u_{-i} \in \mathcal{U}^{n-1}} u_i(f_i(u_i, u_{-i})) = u_i(q, 0) \geq \sup_{u_{-i} \in \mathcal{U}^{n-1}} u_i(f_i(u_i', u_{-i})).$$

By Lemma 3, there is $u_{-i} \in \mathcal{U}^{n-1}$ such that $x_i^f(u_i', u_{-i}) = \mathbf{0}$. By Lemma 1, $f_i(u_i', u_{-i}) =$

$(\mathbf{0}, 0)$. Thus, by *individual rationality*,

$$\inf_{u'_{-i} \in \mathcal{U}^{n-1}} u_i(f_i(u_i, u'_{-i})) \geq u_i(\mathbf{0}, 0) = u_i(f_i(u'_i, u_{-i})). \tag{4}$$

Also, by *individual rationality* and $f_i(u'_i, u_{-i}) = (\mathbf{0}, 0)$,

$$\inf_{u'_{-i} \in \mathcal{U}^{n-1}} u_i(f_i(u'_i, u'_{-i})) = u_i(\mathbf{0}, 0) = u_i(f_i(u'_i, u_{-i})). \tag{5}$$

Combining (4) and (5), we get

$$\inf_{u'_{-i} \in \mathcal{U}^{n-1}} u_i(f_i(u_i, u'_{-i})) \geq \inf_{u'_{-i} \in \mathcal{U}^{n-1}} u_i(f_i(u'_i, u'_{-i})),$$

as desired. ■

A.2 The "only if" part

Next, we show the "only if" part. We show the contrapositive. Suppose that for some $i \in N$ and some $u_i \in \mathcal{U}$, we have $\sup_{u_{-i} \in \mathcal{U}^{n-1}} u_i(f_i(u_i, u_{-i})) \neq u_i(q, 0)$. By *no subsidy*, for each $u_{-i} \in \mathcal{U}^{n-1}$, $f_i(u_i, u_{-i}) \in X \times \mathbb{R}_+$, and so $u_i(f_i(u_i, u_{-i})) \leq u_i(q, 0)$. Thus, we have $\sup_{u_{-i} \in \mathcal{U}^{n-1}} u_i(f_i(u_i, u_{-i})) < u_i(q, 0)$. By continuity of u_i, there is $\varepsilon \in \mathbb{R}_{++}$ sufficiently close to zero such that

$$\sup_{u_{-i} \in \mathcal{U}^{n-1}} u_i(f_i(u_i, u_{-i})) < u_i(q, \varepsilon).$$

By richness, there is $u'_i \in \mathcal{U} \cap \mathcal{U}^Q$ such that $v'_i(q) < \varepsilon$. Let $j \in N \backslash \{i\}$. By richness and object monotonicity of u'_i, there is $u_j \in \mathcal{U} \cap \mathcal{U}^Q$ such that for each pair $x_i, x'_i \in X$ with $x_i > x'_i$, $v_j(q) < v'_i(x_i) - v'_i(x'_i)$. For each $k \in N \backslash \{i, j\}$, let $u_k = u_j$. By *efficiency*, $x_i(u'_i, u_{-i}) = q$. By *individual rationality*, $t^f_i(u'_i, u_{-i}) \leq v'_i(q) < \varepsilon$. Thus,

$$\sup_{u'_{-i} \in \mathcal{U}^{n-1}} u_i(f_i(u_i, u'_{-i})) < u_i(q, \varepsilon) < u_i(f_i(u'_i, u_{-i})) \leq \sup_{u'_{-i} \in \mathcal{U}^{n-1}} u_i(f_i(u'_i, u'_{-i})).$$

Thus, f is *obviously manipulable*. ■

B Proof of Proposition 1

Let f be a truncated pay as bid rule on $\mathcal{U}^n$. Note that *no subsidy* follows from the definition of the rule. Thus, we show the other two properties.

Individual rationality. Let $u \in \mathcal{U}^n$ and $i \in N$. Suppose $u_i(\mathbf{0}, 0) \geq u_i(x_i^f(u), b_i)$. Then, by the definition of the valuation, we have $u_i(x_i^f(u), V_i(x_i^f(u), (\mathbf{0}, 0))) = u_i(\mathbf{0}, 0)$. By $t_i^f(u) = \min\{V_i(x_i^f(u), (\mathbf{0}, 0)), b_i\} \leq V_i(x_i^f(u), (\mathbf{0}, 0))$, $u_i(f_i(u)) \geq u_i(x_i^f(u), V_i(x_i^f(u), (\mathbf{0}, 0)))$. Combining these, we get $u_i(f_i(u)) \geq u_i(\mathbf{0}, 0)$. Suppose instead $u_i(x_i^f(u), b_i) > u_i(\mathbf{0}, 0)$, then $V_i(x_i^f(u), (\mathbf{0}, 0)) = \infty$. Thus, we have $t_i^f(u) = \min\{V_i(x_i^f(u), (\mathbf{0}, 0)), b_i\} = b_i$. Thus, we get $u_i(f_i(u)) = u_i(x_i^f(u), b_i) > u_i(\mathbf{0}, 0)$.

Efficiency. Let $u \in \mathcal{U}^n$. Let $z = (z_i)_{i \in N} = (x_i, t_i)_{i \in N} \in Z$ be an allocation such that for each $i \in N$, $u_i(z_i) \geq u_i(f_i(u))$. To show that z does not Pareto dominate $f(u)$ for u, we need to show that $\sum_{i \in N} t_i \leq \sum_{i \in N} t_i^f(u)$. The proof consists of several claims.

Claim 1. *For each $i \in N$, $t_i \leq V_i(x_i, f_i(R))$.*

Proof. Let $i \in N$. Suppose $u_i(f_i(u)) \geq u_i(x_i, b_i)$. By the definition of the valuation, we have $u_i(f_i(u)) = u_i(x_i, V_i(x_i, f_i(u)))$. Thus, by $u_i(z_i) \geq u_i(f_i(u))$, $u_i(z_i) \geq u_i(x_i, V_i(x_i, f_i(u)))$. By money monotonicity, this implies that $t_i \leq V_i(x_i, f_i(u))$. Suppose instead $u_i(f_i(u)) < u_i(x_i, b_i)$. Then, $V_i(x_i, f_i(u)) = \infty$, and so $t_i < V_i(x_i, f_i(u))$. □

Claim 2. *For each $i \in N$, $t_i \leq b_i$.*

Proof. Let $i \in N$. Recall $u_i(z_i) \geq u_i(f_i(u))$. By *individual rationality*, $u_i(f_i(u)) \geq u_i(\mathbf{0}, 0)$. By $b_i > 0$, $u_i(\mathbf{0}, 0) > -\infty$. Combining these, we get $u_i(z_i) > -\infty$. Thus, $t_i \leq b_i$. □

Claim 3. *For each $i \in N$, $V_i(x_i, f_i(R)) \leq V_i(x_i, (\mathbf{0}, 0))$.*

Proof. Let $i \in N$. Suppose $u_i(\mathbf{0}, 0) \geq u_i(x_i, b_i)$. By *individual rationality*, $u_i(f_i(u)) \geq u_i(\mathbf{0}, 0)$. Combining these, we get $u_i(f_i(u)) \geq u_i(x_i, b_i)$. Thus, by the definition of valuation, we have $u_i(\mathbf{0}, 0) = u_i(x_i, V_i(x_i, (\mathbf{0}, 0)))$ and $u_i(f_i(u)) = u_i(x_i, V_i(x_i, f_i(u)))$. Thus, by *individual rationality*,

$$u_i(x_i, V_i(x_i, f_i(u))) = u_i(f_i(u)) \geq u_i(\mathbf{0}, 0) = u_i(x_i, V_i(x_i, (\mathbf{0}, 0))).$$

Thus, by money monotonicity, $V_i(x_i, f_i(R)) \leq V_i(x_i, (\mathbf{0}, 0))$. Suppose instead $u_i(\mathbf{0}, 0) < u_i(x_i, b_i)$. Then, $V_i(x_i, (\mathbf{0}, 0)) = \infty$, and so $V_i(x_i, f_i(R)) \leq V_i(x_i, (\mathbf{0}, 0))$. □

Claim 4. *For each $i \in N$, $t_i \leq \tilde{v}_i(x_i)$.*

Proof. Let $i \in N$. We have

$$t_i \leq \min\Big\{V_i(x_i, f_i(u)), b_i\Big\} \leq \min\Big\{V_i(x_i, (\mathbf{0}, 0)), b_i\Big\} = \tilde{v}_i(x_i),$$

where the first inequality follows from Claims 1 and 2, and the second one from Claim 3. □

Now, we complete the proof. We have

$$\sum_{i \in N} t_i \leq \sum_{i \in N} \tilde{v}_i(x_i) \leq \sum_{i \in N} \tilde{v}_i(x_i^f(u)) = \sum_{i \in N} t_i^f(u),$$

where the first inequality follows from Claim 4, the second one from the definition of an object allocation rule, and the equality from the definition of a payment rule. Thus, we get $\sum_{i \in N} t_i \leq \sum_{i \in N} t_i^f(u)$, and so z does not Pareto dominate $f(u)$ for u. ■

C Proof of Theorem 2

Let f be a modified pay as bid rule on $\mathcal{U}^n$. Note that *no subsidy* follows from the definition of the rule. Thus, we show that f satisfies the other three properties.

Efficiency. Let $u \in \mathcal{U}^n$. By Proposition 1, if $f(u)$ is an outcome allocation of a truncated pay as bid rule for u, then $f(u)$ is efficient for u. Suppose that there is $i \in N$ such that $f_i(u) = (q, 0)$, and for each $j \in N \backslash \{i\}$, $f_j(u) = (\mathbf{0}, 0)$. By the definition of the rule, for each $x_i \in X \backslash \{q\}$

$$-V_i(x_i, (q, 0)) > (n-1) \max_{j \in N \backslash \{i\}} \tilde{v}_j(q). \tag{1}$$

Let $z = (z_j)_{j \in N} = (x_j, t_j)_{j \in N} \in Z$ be an allocation such that for each $j \in N$, $u_j(z_j) \geq u_j(f_j(u))$. To show that z does not Pareto dominate $f(u)$ for u, we show that $\sum_{j \in N} t_j^f(u) \geq \sum_{j \in N} t_j$. By object monotonicity and $b_i \in \mathbb{R}_{++}$, $u_i(q, 0) > u_i(x_i, b_i)$. Then,

$$u_i(z_i) \geq u_i(f_i(u)) = u_i(q, 0) = u_i(x_i, V_i(x_i, (q, 0))),$$

where the last equality follows from $u_i(q,0) > u_i(x_i,b_i)$ and the definition of the valuation. Thus, by money monotonicity,

$$t_i \leq V_i(x_i,(q,0)). \tag{2}$$

For each $j \in N\backslash\{i\}$, $u_j(z_j) \geq u_j(f_j(u)) = u_j(\mathbf{0},0)$, which implies

$$t_j \leq \tilde{v}_j(x_j). \tag{3}$$

Then, by (2) and (3),

$$\sum_{j\in N} t_j \leq V_i(x_i,(q,0)) + \sum_{j\in N\backslash\{i\}} \tilde{v}_j(x_j). \tag{4}$$

If $x_i = q$, then for each $j \in N\backslash\{i\}$, $x_j = \mathbf{0}$, and so $\tilde{v}_j(x_j) = 0$. Thus,

$$\sum_{j\in N} t_j \leq V_i(q,(q,0)) = 0 \leq \sum_{j\in N} t^f_j(u),$$

where the first inequality follows from (4), and the last inequality from *no subsidy*. If $x_i \neq q$, then

$$\sum_{j\in N} t_j \leq V_i(x_i,(q,0)) + (n-1)\max_{j\in N\backslash\{i\}} \tilde{v}_j(q) < 0 \leq \sum_{j\in N} t^f_j(u),$$

where the first inequality follows from (4) and object monotonicity, the second one from (1), and the last one from *no subsidy*. Thus, in either case, $\sum_{j\in N} t_j \leq \sum_{j\in N} t^f_j(u)$, and so z does not Pareto dominate $f(u)$ for u.

Individual rationality. Let $u \in \mathcal{U}^n$ and $i \in N$. If $f_i(u)$ is an outcome bundle of agent i for u under a truncated pay as bid rule, then by Proposition 1, $u_i(f_i(u)) \geq u_i(\mathbf{0},0)$. Otherwise, by the definition of the rule, $f_i(u) \in \{(q,0),(\mathbf{0},0)\}$. Thus, $u_i(f_i(u)) \geq u_i(\mathbf{0},0)$.

Non-obvious manipulability. Let $i \in N$ and $u_i \in \mathcal{U}$. By object monotonicity and $b_i \in \mathbb{R}_{++}$, for each $x_i \in X\backslash\{\mathbf{0}\}$, $\tilde{v}_i(x_i) > 0$. Let $x_i \in X\backslash\{q\}$. Again, by object monotonicity and $b_i \in \mathbb{R}_{++}$, $u_i(q,0) > u_i(x_i,b_i)$. Thus, by the definition of the valuation and object monotonicity,

$$u_i(x_i, V_i(x_i,(q,0))) = u_i(q,0) > u_i(x_i,0).$$

Thus, by money monotonicity, $V_i(x_i,(q,0)) < 0$, or equivalently, $-V_i(x_i,(q,0)) > 0$. Thus, by richness, there is $u_{-i} \in (\mathcal{U} \cap \mathcal{U}^Q)^{n-1}$ such that for each $x_i \in X \backslash \{\mathbf{0}\}$ and each $x_i' \in X \backslash \{q\}$, $(n-1)\max_{j \in N \backslash \{i\}} \tilde{v}_j(q) < \min\{\tilde{v}_i(x_i), -V_i(x_i',(q,0))\}$. Thus, by the definition of the rule, $f_i(u) = (q,0)$. Note that by *no subsidy*, for each $u'_{-i} \in \mathcal{U}^{n-1}$, $f_i(u_i,u'_{-i}) \in X \times \mathbb{R}_+$. Thus, by money monotonicity and object monotonicity,

$$\sup_{u'_{-i} \in \mathcal{U}^{n-1}} u_i(f_i(u_i,u'_{-i})) = u_i(f_i(u)) = u_i(q,0).$$

Note that we have already shown that f satisfies *efficiency*, *individual rationality*, and *no subsidy*. Thus, by Theorem 1, f is *non-obviously manipulable*. ■

D Proof of Proposition 2

In this section, we prove Proposition 2.

D.1 Proposition 2 (i)

Let f be a truncated Vickrey rule on $(\mathcal{U}^{IE})^n$. We have already shown in Example 1 that it violates *efficiency*. Thus, we show that it is *non-obviously manipulable*. Since the proof is similar to that of the "if" part of Theorem 1, we only give an informal sketch of the proof. Let $i \in N$ and $u_i \in \mathcal{U}^{IE}$. Then, we can choose $u_{-i} \in (\mathcal{U}^{IE})^{n-1}$ whose valuations of q at $(\mathbf{0},0)$ are arbitrarily small. Thus, by the definition of a truncated Vickrey rule, the best case utility from the truth-telling is $u_i(q,0)$, and so applying the same discussion as the "if" part of Theorem 1, we can conclude that agent i cannot improve the best case utility by reporting a false utility function. Also, for each $u_i' \in \mathcal{U}^{IE}$, we can choose $u'_{-i} \in (\mathcal{U}^{IE})^{n-1}$ whose valuations at $(\mathbf{0},0)$ are so large compared with u_i' that $f_i(u_i',u'_{-i}) = (\mathbf{0},0)$. It is straightforward to verify that a truncated Vickrey rule on any domain $\mathcal{U}$ satisfies *individual rationality* and *no subsidy*. Thus, as in the proof of the "if" part of Theorem 1, we can show that the worst case utility from any report u_i' is $u_i(\mathbf{0},0)$, and so agent i cannot improve the worst case utility by misreporting his utility function. ■

D.2 Proposition 2 (ii)

Le (2018) provides an example which shows that a truncated Vickrey rule on $(\mathcal{U}^{HB})$ violates *efficiency*. Thus, we here show that if $\sum_{a \in M} q^a = 1$, then any truncated Vickrey rule on $(\mathcal{U}^{HB})^n$ is *non-obviously manipulable*, while if $\sum_{a \in M} q^a \geq 2$, then it is *obviously manipulable*. Let f be a truncated Vickrey rule on $(\mathcal{U}^{HB})$.

Suppose $\sum_{a \in M} q^a = 1$. Let $i \in N$ and $u_i \in \mathcal{U}^{HB}$. By object monotonicity and $b_i \in \mathbb{R}_{++}$, $\tilde{v}_i(1) > 0$. Thus, we can choose $u_{-i} \in (\mathcal{U}^{HB})^{n-1}$ such that $\max_{j \in N \setminus \{i\}} \tilde{v}_j(1) < \tilde{v}_i(1)$, and $\max_{j \in N \setminus \{i\}} \tilde{v}_j(1)$ can be arbitrarily small. Thus, the best case utility of agent i from the truth-telling is $u_i(1, 0)$, and so he cannot improve the best case utility by misrepresenting his utility function. Also, by the same discussion as in the proof of Proposition 2 (i), he cannot improve the worst case utility from the truth-telling by misrepresenting his utility function. Thus, f is *non-obviously manipulable*.

Next, suppose $\sum_{a \in M} q^a \geq 2$. For simplicity, let $n = 2$, $a = 1$, and $q = 2$, but the following discussion extends to a more general setting. Let $u_1 \in \mathcal{U}^{HB}$ be such that $v_1(1) = 1$, $v_1(2) = 2$, and $b_1 = 1$. Then, $\tilde{v}_1(2) = \tilde{v}_1(1) = 1$. Then, for each $u_2 \in \mathcal{U}^{HB}$, by $\tilde{v}_2(1) > 0$, $x_1^f(u_1, u_2) \leq 1$. Thus, $\sup_{u_2 \in \mathcal{U}} u_1(f_1(u_1, u_2)) = u_1(1, 0) = 1$. Let $u_1' \in \mathcal{U}^{HB}$ be such that $v_1'(1) = 1$, $v_1'(2) = 2$, and $b_1' = \infty$. Then, $\sup_{u_2 \in \mathcal{U}} u_1(f_1(u_1', u_2)) = u_1(2, 0) = 2$. Thus,

$$\sup_{u_2 \in \mathcal{U}} u_1(f_1(u_1', u_2)) = 2 > 1 = \sup_{u_2 \in \mathcal{U}} u_1(f_1(u_1, u_2)).$$

Thus, f is *non-obviously manipulable*. ■

E Proof of Proposition 3

In this section, we prove Proposition 3. Given $u \in \mathcal{U}^n$ and $i \in N$, let $Q = \sum_{a \in M} q^a$ and $\tilde{V}_{-i} = \max_{j \in N \setminus \{i\}} \tilde{v}_j(q)$.

First, we show that if for each $x_i \in X \setminus \{\mathbf{0}\}$, $\tilde{v}_i(x_i) > Q\tilde{V}_{-i}$, and for each $x_i \in X \setminus \{q\}$, $V_i(x_i, (q, Q\tilde{V}_{-i})) < 0$, then an allocation $z = (z_j)_{j \in N} \in Z$ such that $z_i = (q, Q\tilde{V}_{-i})$, and for each $j \in N \setminus \{i\}$, $z_j = (\mathbf{0}, 0)$ is a Warlasian equilibrium allocation for u. Let $p \in \mathbb{R}^m_+$ be a price vector such that $p^a = \tilde{V}_{-i}$ for each $a \in M$. Then, $p \cdot q = Q\tilde{V}_{-i}$. For each $x_i \in X \setminus \{q\}$, by $V_i(x_i, (q, Q\tilde{V}_{-i})) < 0 \leq p \cdot x_i$, we have $u_i(q, p \cdot q) = u_i(q, Q\tilde{V}_{-i}) > u_i(x_i, p \cdot x_i)$. For each $j \in N \setminus \{i\}$ and each $x_j \in X \setminus \{\mathbf{0}\}$, by $\tilde{v}_j(x_j) \leq \tilde{V}_{-i} \leq (\sum_{a \in M} q^a)\tilde{V}_{-i} = p \cdot x_j$, we have

$u_j(\mathbf{0}, 0) \geq u_j(x_j, p \cdot x_j)$. Thus, z is a Warlasian equilibrium allocation for u.

Then, we construct a Warlasian rule that will satisfy *non-obvious manipulability*. Let f be a rule on $\mathcal{U}^n$ such that for each $u \in \mathcal{U}^n$, if there is $i \in N$ such that for each $x_i \in X \backslash \{\mathbf{0}\}$, $\tilde{v}_i(x_i) > Q\tilde{V}_{-i}$, and for each $x_i \in X \backslash \{q\}$, $V_i(x_i, (q, Q\tilde{V}_{-i})) < 0$, then $f_i(u) = (q, Q\tilde{V}_{-i})$ and for each $j \in N \backslash \{i\}$, $f_j(u) = (\mathbf{0}, 0)$, and otherwise, $f(u)$ is a Warlasian equilibrium allocation for u. Note that by the assumption that for each $u \in \mathcal{U}^n$, there is a Warlasian equilibrium allocation for u, the rule f is well-defined. Further, by the discussion in the previous paragraph, f is a Warlasian rule.

We finally show that f is *non-obviously manipulable*. Let $i \in N$ and $u_i \in \mathcal{U}$. Note that any Warlasian rule satisfies *no subsidy*, and so does f. Thus, for each $u_{-i} \in \mathcal{U}^{n-1}$, by *no subsidy* of f, $f_i(u_i, u_{-i}) \in X \times \mathbb{R}_+$. Thus, by money monotonicity and object monotonicity, for each $u_{-i} \in \mathcal{U}^{n-1}$, $u_i(f_i(u_i, u_{-i})) \leq u_i(q, 0)$. Let $\varepsilon \in \mathbb{R}_{++}$. Note that by object monotonicity of u_i and $b_i \in \mathbb{R}_{++}$, for each $x_i \in X \backslash \{q\}$, $V_i(x_i, (q, 0)) < 0$. Thus, by continuity of u_i and $b_i \in \mathbb{R}_{++}$, we can choose $\delta \in \mathbb{R}_{++}$ sufficiently close to 0 such that $\delta < b_i$, $u_i(q, \delta) + \varepsilon > u_i(q, 0)$, and for each $x_i \in X \backslash \{q\}$, $V_i(x_i, (q, \delta)) < 0$. Again, by object monotonicity and $b_i \in \mathbb{R}_{++}$, for each $x_i \in X \backslash \{\mathbf{0}\}$, $\tilde{v}_i(x_i) > 0$. Thus, by richness, there is $u_{-i} \in \mathcal{U}^{n-1}$ such that for each $x_i \in X \backslash \{\mathbf{0}\}$, $Q\tilde{V}_{-i} < \min\{\delta, \tilde{v}_i(x_i)\}$. For each $x_i \in X \backslash \{q\}$, by $Q\tilde{V}_{-i} < \delta < b_i$, $u_i(q, Q\tilde{V}_{-i}) > u_i(q, \delta) > u_i(x_i, b_i)$, which implies $V_i(x_i, (q, Q\tilde{V}_{-i})) < V_i(x_i, (q, \delta)) < 0$. Thus, by the definition of the rule, $f_i(u) = (q, Q\tilde{V}_{-i})$. By $Q\tilde{V}_{-i} < \delta$ and $u_i(q, \delta) + \varepsilon > u_i(q, 0)$, $u_i(f_i(u)) + \varepsilon > u_i(q, \delta) + \varepsilon > u_i(q, 0)$. Since $\varepsilon \in \mathbb{R}_{++}$ was arbitrary, we have

$$\sup_{u_{-i} \in \mathcal{U}^{n-1}} u_i(f_i(u_i, u_{-i})) = u_i(q, 0).$$

Since any Warlasian rule satisfies *efficiency*, *individual rationality*, and *no subsidy*, so does f. Thus, by Theorem 1, f is *non-obviously manipulable*. ■

References

[1] Ausubel, L. M. (2004), "An efficient ascending-bid auction for multiple objects." *American Economic Review*, 94(5): 1452–1475.

[2] Archbold, T., B. de Keijzer, and C. Ventre (2023a), "Non-obvious manipulability for single-parameter agents and bilateral trade." In: *Proceedings of the 22nd International*

Conference on Autonomous Agents and Multiagent Systems (AAMAS2023).

[3] Archbold, T., B. de Keijzer, and C. Ventre (2023b), "Non-obvious manipulability in extensive-form mechanisms: The revelation principle for single-parameter agents." In: *Proceedings of the 32nd International Joint Conference on Artificial Intelligence (IJCAI-23)*.

[4] Arribillaga, R. P. and A. G. Bonifacio (2024), "Obvious manipulations of tops-only voting rules." *Games and Economic Behavior*, 143: 12–24.

[5] Arribillaga, R. P. and E. Risma (2023), "Obvious manipulations in matching with and without contracts." Working paper.

[6] Aziz, H. and A. Lam (2021), "Obvious manipulability of voting rules." In: Fotakis, D. and D. Riós Insua (eds), *Algorithmic Decision Theory*: 179–193, Springer, Cham.

[7] Baisa, B. (2017), "Auction design without quasilinear preferences." *Theoretical Economics*, 12(1): 53–78.

[8] Baisa, B. (2018), "A detail-free and efficient auction for budget constrained bidders." Working paper.

[9] Baisa, B. (2020), "Efficient multi-unit auctions for normal goods." *Theoretical Economics*, 15(1): 361–413.

[10] Baisa, B. and S. Essig Aberg (2021), "Quantifying the inefficiency of multi-unit auctions for normal goods." Working paper.

[11] Baldwin, E., O. Edhan, R. Jagadeesan, P. Klemperer, and A. Teytelboym (2023), "The equilibrium existence duality: Equilibrium with indivisibilities and income effects." *Journal of Political Economy*, 131(6): 1440–1476.

[12] Bulow, J., J. Levin, and P. Milgrom (2017), "Winning play in spectrum auctions." In: Bichler, M. and J. K. Goeree (eds), *Handbook of spectrum auction design*, Chapter 31, pp. 689–712, Cambridge University Press.

[13] Chew, S. H. and S. Serizawa (2007), "Characterizing the Vickrey combinatorial auction by induction." *Economic Theory*, 33(2): 393–406.

[14] Demange, G. and D. Gale (1985), "The strategy structure of two-sided matching markets." *Econometrica*, 53(4): 873–888.

[15] Dobzinski, S., R. Lavi, and N. Nisan (2012), "Multi-unit auctions with budget limits." *Games and Economic Behavior*, 74(2): 486–503.

[16] Gul, F. and E. Stacchetti (1999), "Warlasian equilibrium with gross substitutes." *Journal of Economic Theory*, 87(1): 95–124.

[17] Hafalir, I. E., R. Ravi, and A. Sayedi (2012), "A near Pareto optimal auction with budget constraints." *Games and Economic Behavior*, 74(2): 699–708.

[18] Holmström, B. (1979), "Groves' scheme on restricted domains." *Econometrica*, 47(5): 1137–1144.

[19] Kazumura, T. (2022), "When can we design efficient and strategy-proof rules in package assignment problems?" Kyoto University, Graduate School of Economics Discussion Paper Series No. E-21-008.

[20] Kazumura, T. and S. Serizawa (2016), "Efficiency and strategy-proofness in object assignment problems with multi-demand preferences." *Social Choice and Welfare*, 47(3): 633–663.

[21] Kelso, A. S. and V. P. Crawford (1982), "Job matching, coalition formation, and gross substitutes." *Econometrica*, 50(6): 1483–1504.

[22] Lavi, R. and M. May (2012), "A note on the incompatibility of strategy-proofness and Pareto-optimality in quasi-linear settings with public budgets."

[23] Le, P. (2018), "Pareto optimal budgeted combinatorial auctions." *Theoretical Economics*, 13(2): 831–868.

[24] Li, S. (2017), "Obviously strategy-proof mechanisms." *American Economic Review*, 107(11): 3257–3287.

[25] Malik, K. and D. Mishra (2021), "Pareto efficient combinatorial auctions: Dichotomous preferences without quasilinearity." *Journal of Economic Theory*, 191: 105128.

[26] Morimoto, S. and S. Serizawa (2015), "Strategy-proofness and efficiency with non-quasi-linear preferences: A characterization of minimum price Warlasian rule." *Theoretical Economics*, 10(2): 445–487.

[27] Ortega, J. and E. Segal-Halevi (2022), "Obvious manipulations in cake-cutting." *Social Choice and Welfare*, 59: 969–988.

[28] Psomas, A. and P. Verma (2022), "Fair and efficient allocations without obvious manipulations." Working paper.

[29] Rostek, M. and N. Yoder (2020), "Matching with complementary contracts." *Econometrica*, 88(5): 1793–1827.

[30] Saitoh, H. and S. Serizawa (2008), "Vickrey allocation rule with income effect." *Economic Theory*, 35(2): 391–401.

[31] Sakai, T. (2008), "Second price auctions on general preference domains: Two characterizations." *Economic Theory*, 37(2): 347–356.

[32] Shinozaki, H. (2023), "Efficiency and strategy-proofness in multi-unit object allocation problems with non-quasi-linear preferences." *Economics Letters*, 223: 110989.

[33] Shinozaki, H., T. Kazumura, and S. Serizawa (2022), "Multi-unit object allocation problems with money for (non)decreasing incremental willingness to pay: Impossibility and characterization theorems." ISER Discussion paper No. 1097.

[34] Troyan, P. (2022), "Non-obvious manipulability of the rank-minimizing mechanism." Working paper.

[35] Troyan, P. and T. Morrill (2020), "Obvious manipulations." *Journal of Economic Theory*, 185: 104970.

[36] Vickrey, W. (1961), "Counterspeculation, auctions, and competitive sealed tenders." *Journal of Finance*, 16(1): 8–37.